Signal Processing First

Lecture 5

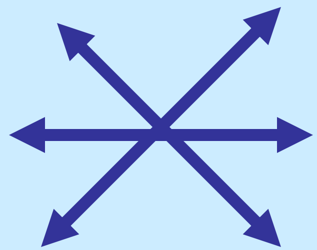
Periodic Signals, Harmonics & Time-Varying Sinusoids

READING ASSIGNMENTS

- This Lecture:
  - Chapter 3, Sections 3-2 and 3-3
  - Chapter 3, Sections 3-7 and 3-8
- Next Lecture:
  - **Fourier Series ANALYSIS**
  - Sections 3-4, 3-5 and 3-6

Problem Solving Skills

- Math Formula
  - Sum of Cosines
  - Amp, Freq, Phase
- Plot & Sketches
  - S(t) versus t
  - Spectrum



- Recorded Signals
  - Speech
  - Music
  - No simple formula
- MATLAB
  - Numerical
  - Computation
  - Plotting list of numbers

LECTURE OBJECTIVES

- Signals with HARMONIC Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

FREQUENCY can change vs. TIME

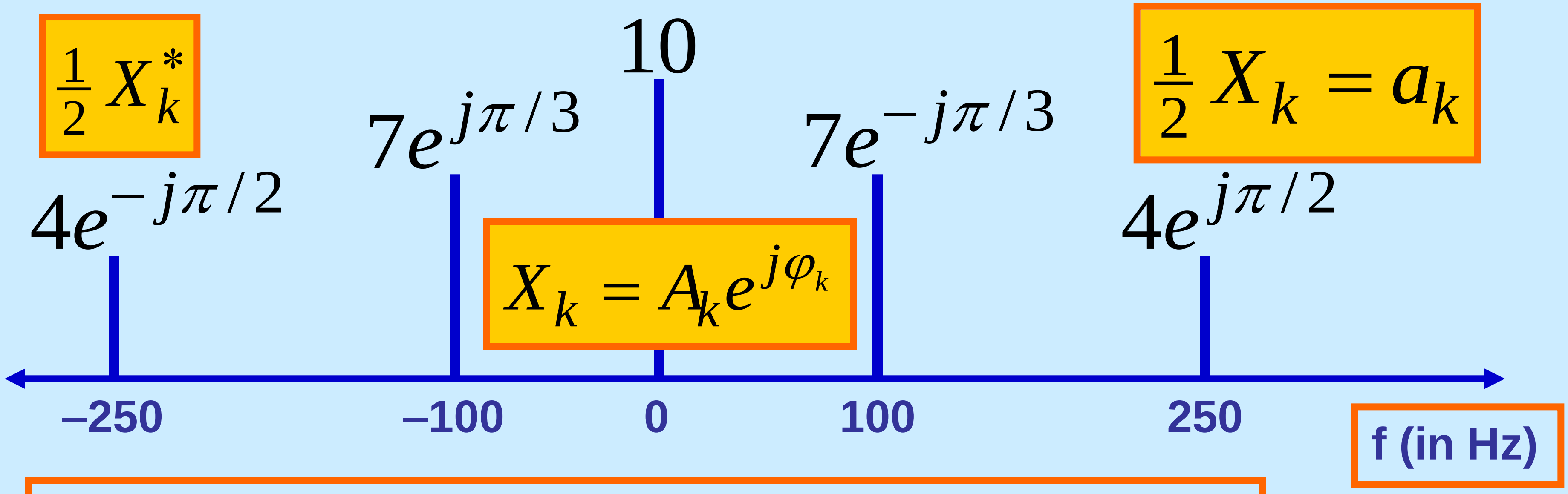
Chirps:

$$x(t) = \cos(\alpha t^2)$$

Introduce Spectrogram Visualization **(specgram.m)** **(plotspec.m)**

SPECTRUM DIAGRAM

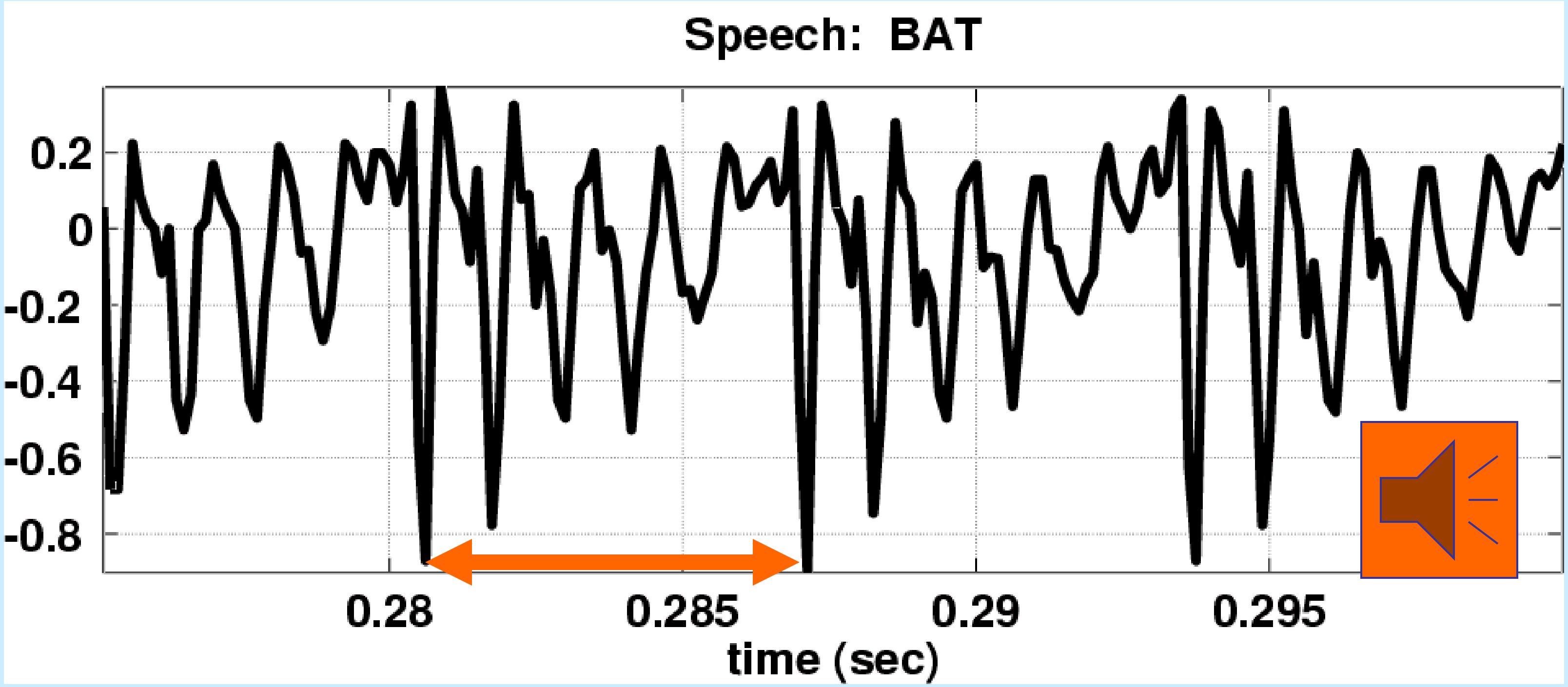
- Recall Complex Amplitude vs. Freq



$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

SPECTRUM for PERIODIC ?

- Nearly Periodic in the Vowel Region
  - Period is (Approximately) T = 0.0065 sec



PERIODIC SIGNALS

- Repeat every T secs
  - Definition

$$x(t) = x(t + T)$$

  - Example:

$$x(t) = \cos(3t)$$

$$T = ?$$

  - Speech can be “quasi-periodic”

Period of Complex Exponential

$$x(t) = e^{j\omega t}$$

$$x(t+T) = x(t)\ ?$$

Definition: Period is T

$$e^{j\omega(t+T)} = e^{j\omega t}$$

$$\Rightarrow e^{j\omega T} = 1 \quad \Rightarrow \omega T = 2\pi k$$

$$e^{j2\pi k} = 1$$

$$\omega = \frac{2\pi k}{T} = \left(\frac{2\pi}{T}\right) k = \omega_0 k$$

k = integer

Harmonic Signal Spectrum

Periodic signal can only have: $f_k = k f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T}$$

$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

Define FUNDAMENTAL FREQ

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \phi_k)$$

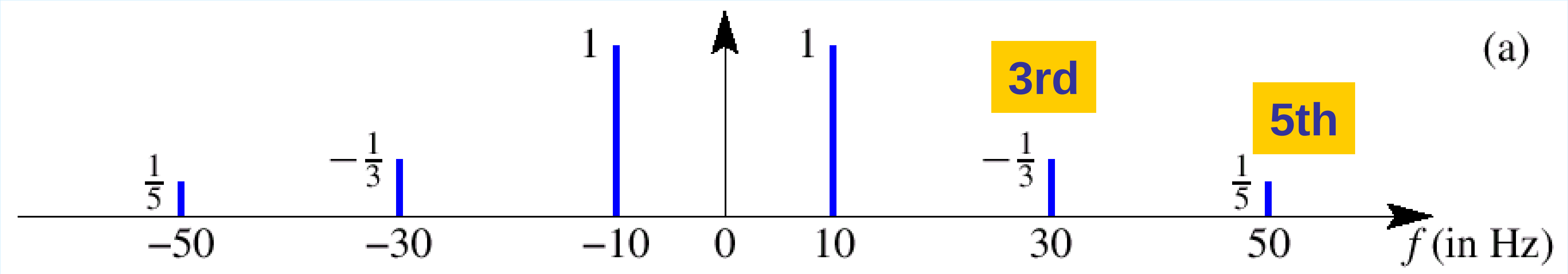
$$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$$

$$f_0 = \frac{1}{T_0}$$

f_0 = fundamental Frequency

T_0 = fundamental Period

Harmonic Signal (3 Freqs)

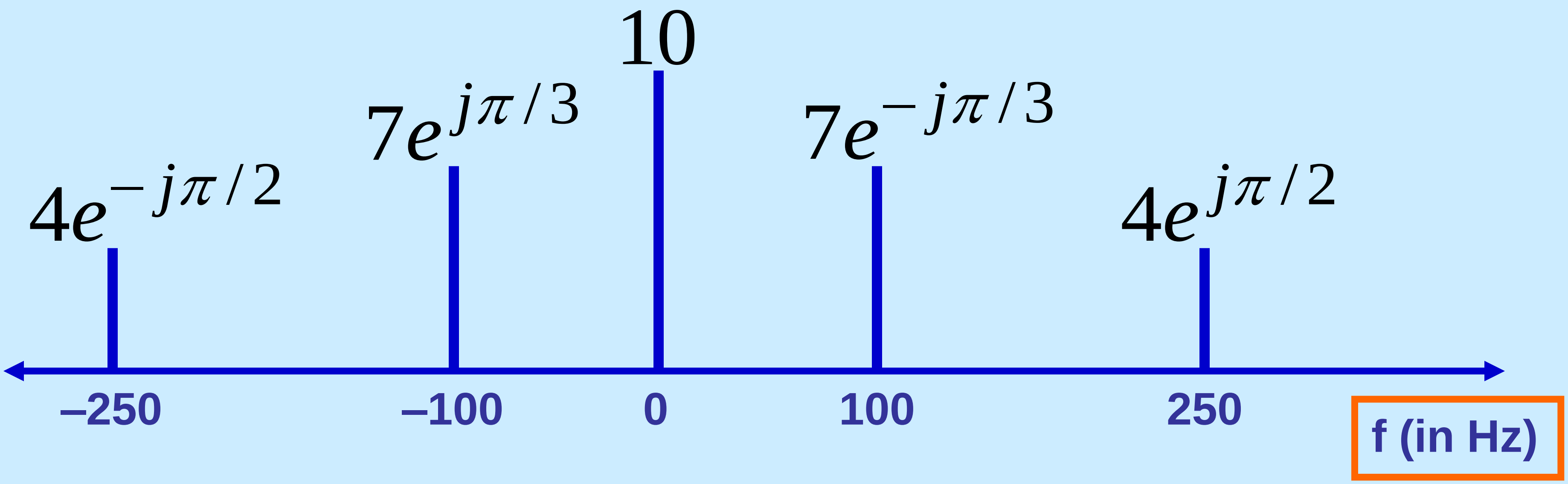


What is the fundamental frequency?

10 Hz

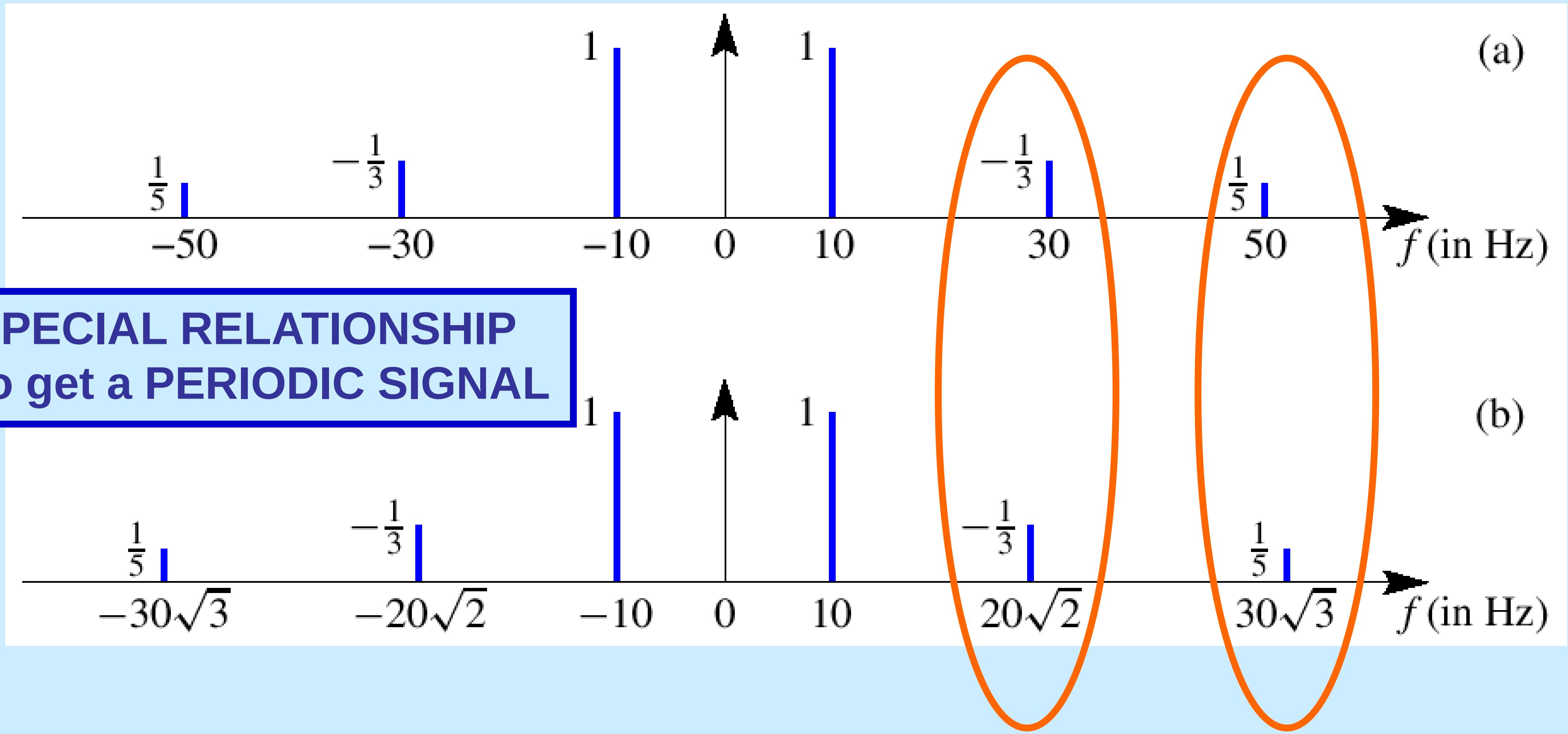
POP QUIZ: FUNDAMENTAL

- Here’s another spectrum:



What is the fundamental frequency?

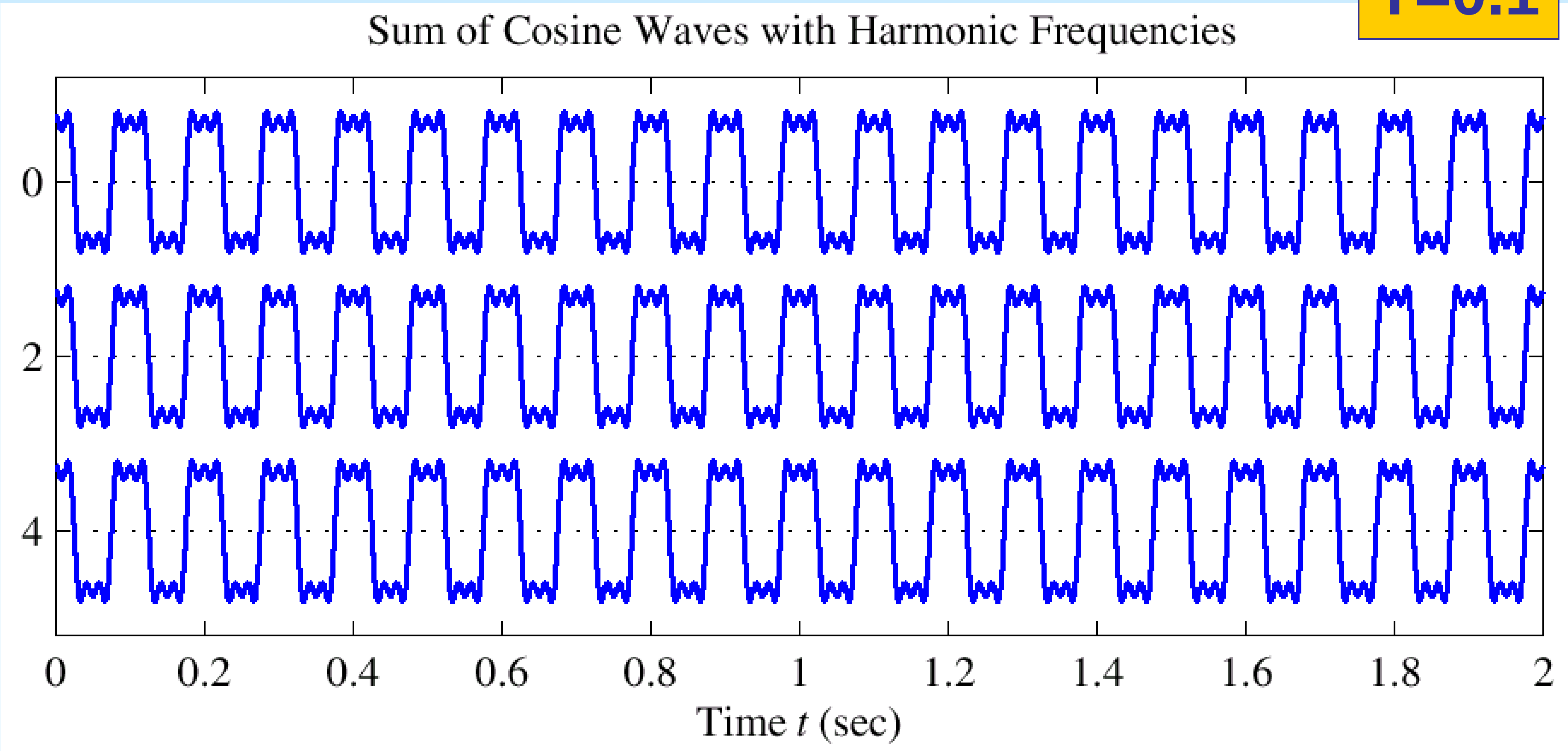
IRRATIONAL SPECTRUM



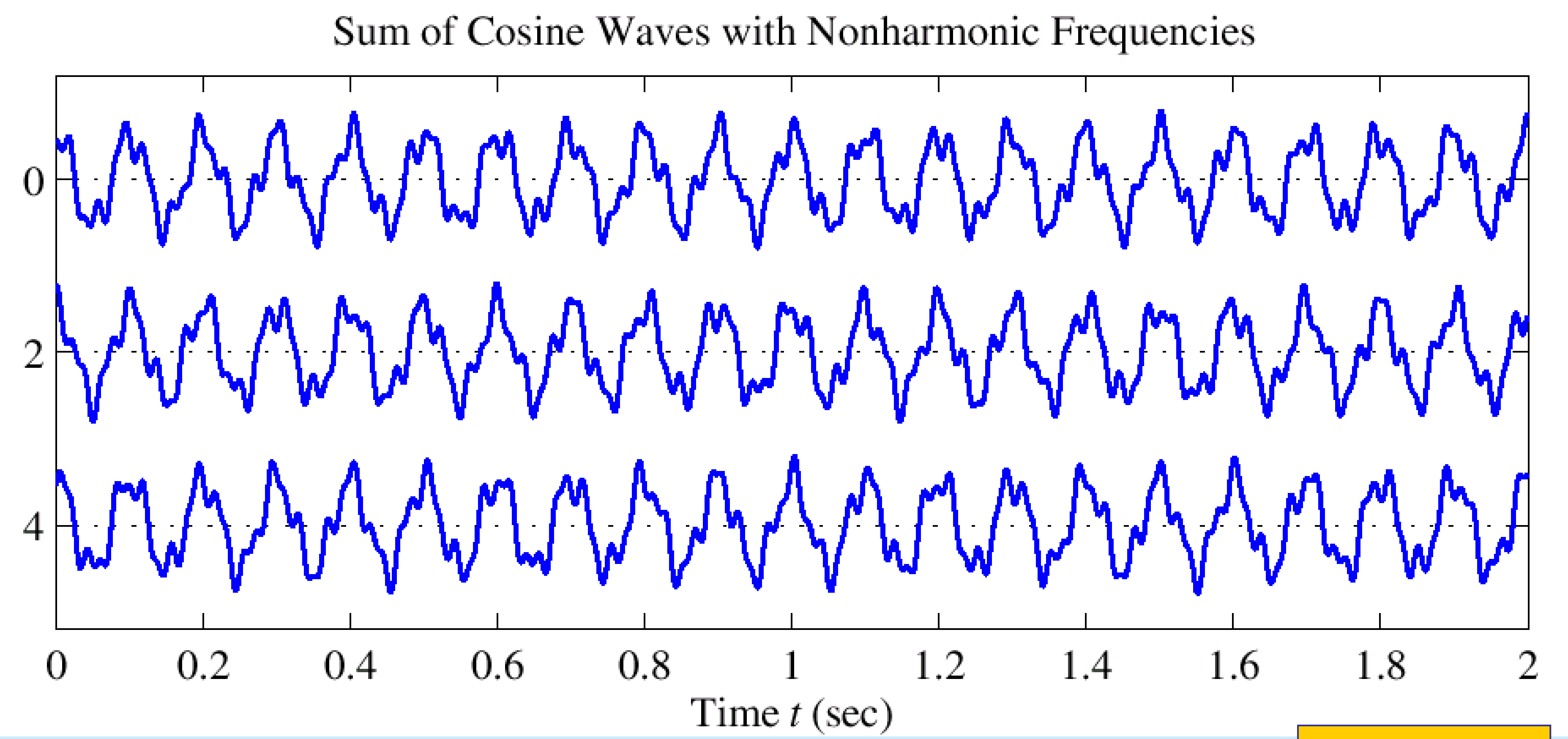
SPECIAL RELATIONSHIP to get a PERIODIC SIGNAL

Harmonic Signal (3 Freqs)

T=0.1



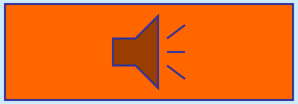
NON-Harmonic Signal

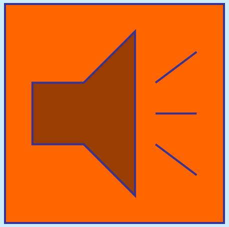


NOT PERIODIC

FREQUENCY ANALYSIS

- **<u>Now, a much HARDER problem</u>**
- Given a recording of a song, have the computer write the music





- Can a machine extract frequencies?
  - Yes, if we COMPUTE the spectrum for $x(t)$
    - During short intervals

Time-Varying FREQUENCIES Diagram

Frequency is the vertical axis

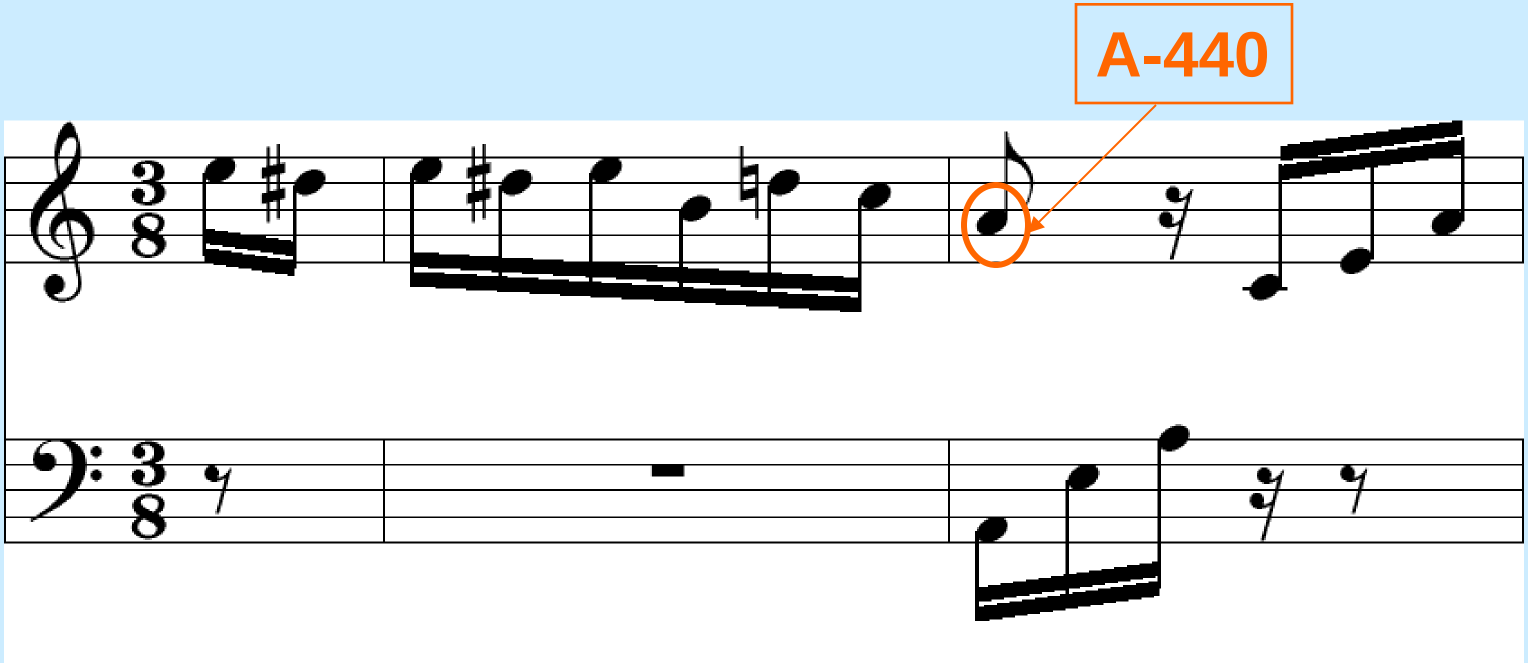
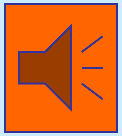


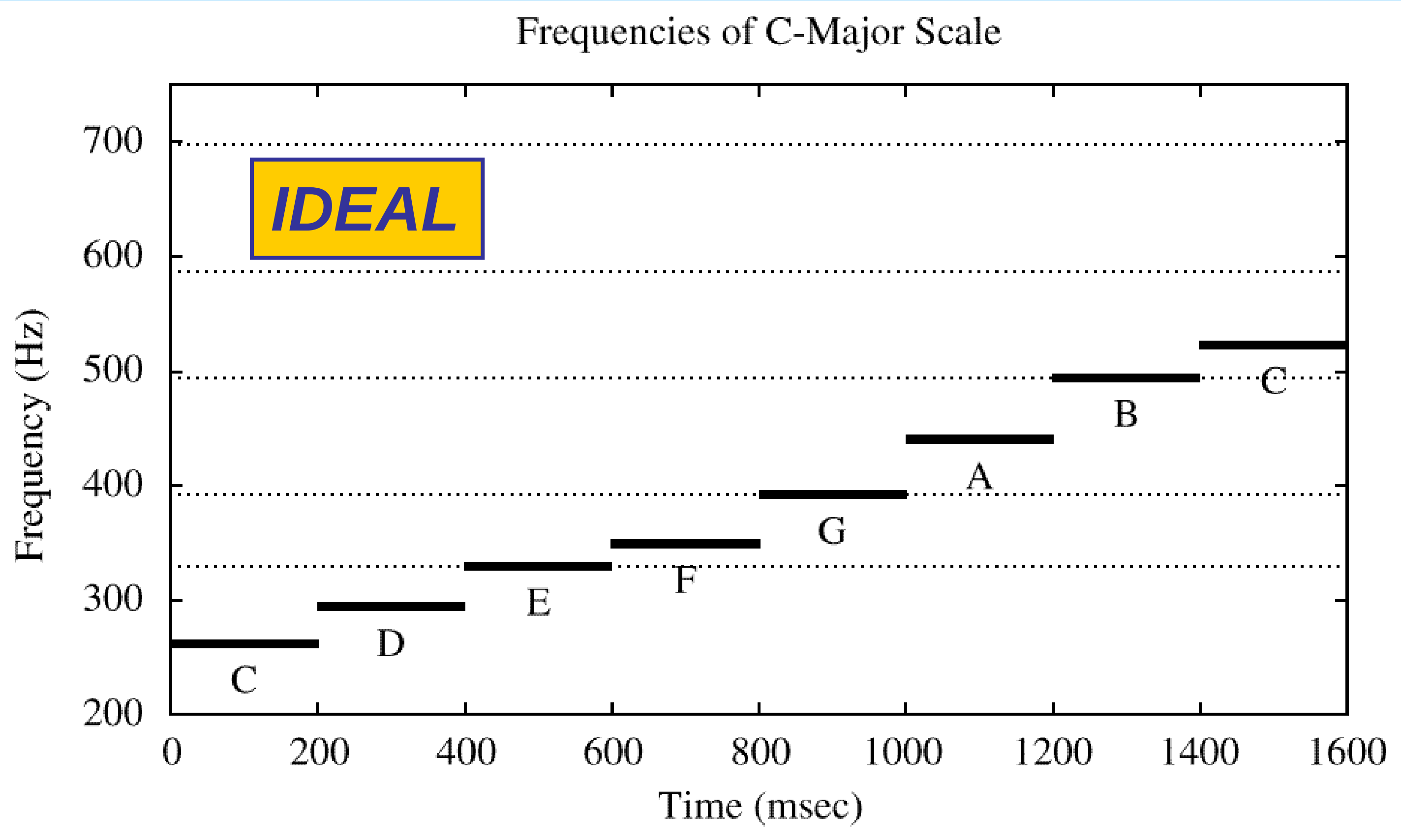
Figure 3.18 Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

SIMPLE TEST SIGNAL

- C-major SCALE: stepped frequencies
  - Frequency is constant for each note



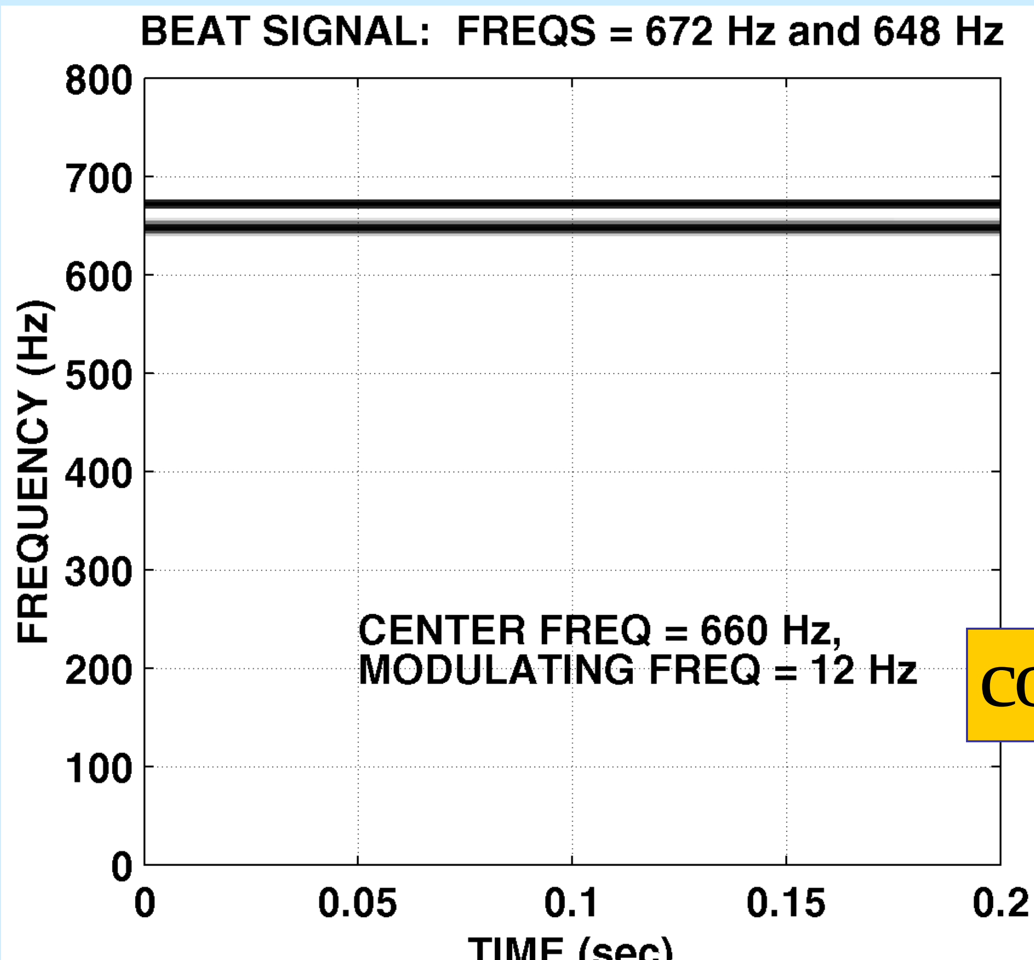


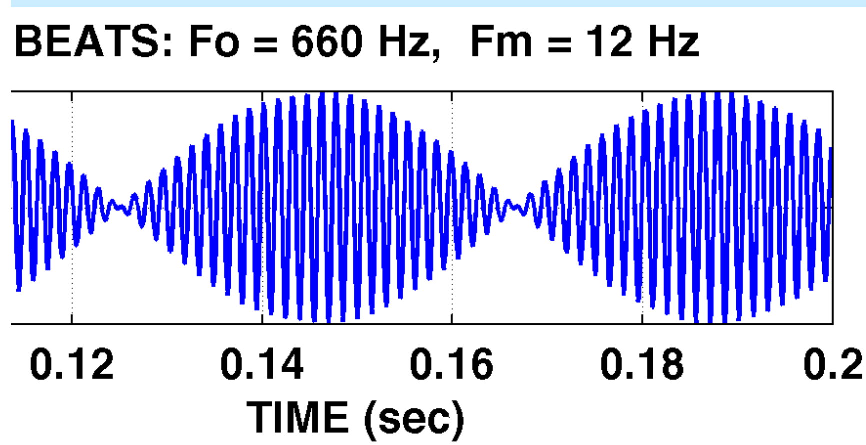
R-rated: ADULTS ONLY

- SPECTROGRAM Tool
  - MATLAB function is **`specgram.m`**
  - SP-First has **`plotspec.m`** & **`spectgr.m`**
- **ANALYSIS** program
  - Takes x(t) as input &
  - Produces spectrum values X_k
  - Breaks x(t) into SHORT TIME SEGMENTS
    - Then uses the FFT (Fast Fourier Transform)

SPECTROGRAM EXAMPLE

- Two **Constant** Frequencies: Beats





$$\cos(2\pi(660)t)\sin(2\pi(12)t)$$

AM Radio Signal

- Same as BEAT Notes

$$\cos(2\pi(660)t)\sin(2\pi(12)t)$$

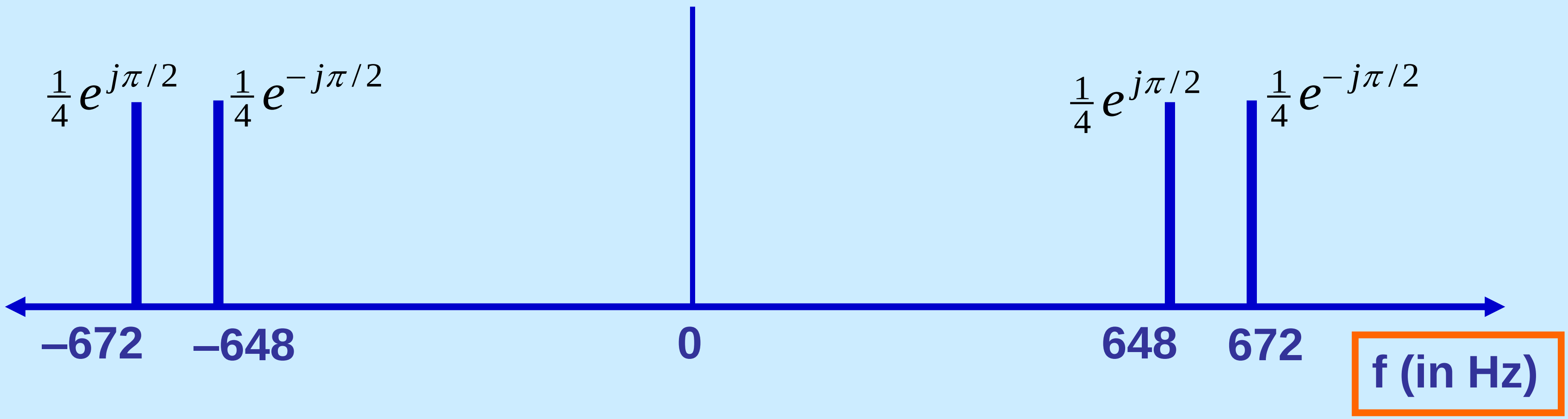
$$\tfrac{1}{2}\left(e^{j2\pi(660)t}+e^{-j2\pi(660)t}\right)\frac{1}{2j}\left(e^{j2\pi(12)t}-e^{-j2\pi(12)t}\right)$$

$$\frac{1}{4j}\left(e^{j2\pi(672)t}-e^{-j2\pi(672)t}-e^{j2\pi(648)t}+e^{-j2\pi(648)t}\right)$$

$$\tfrac{1}{2}\cos(2\pi(672)t-\tfrac{\pi}{2})+\tfrac{1}{2}\cos(2\pi(648)t+\tfrac{\pi}{2})$$

SPECTRUM of AM (Beat)

- 4 complex exponentials in AM:

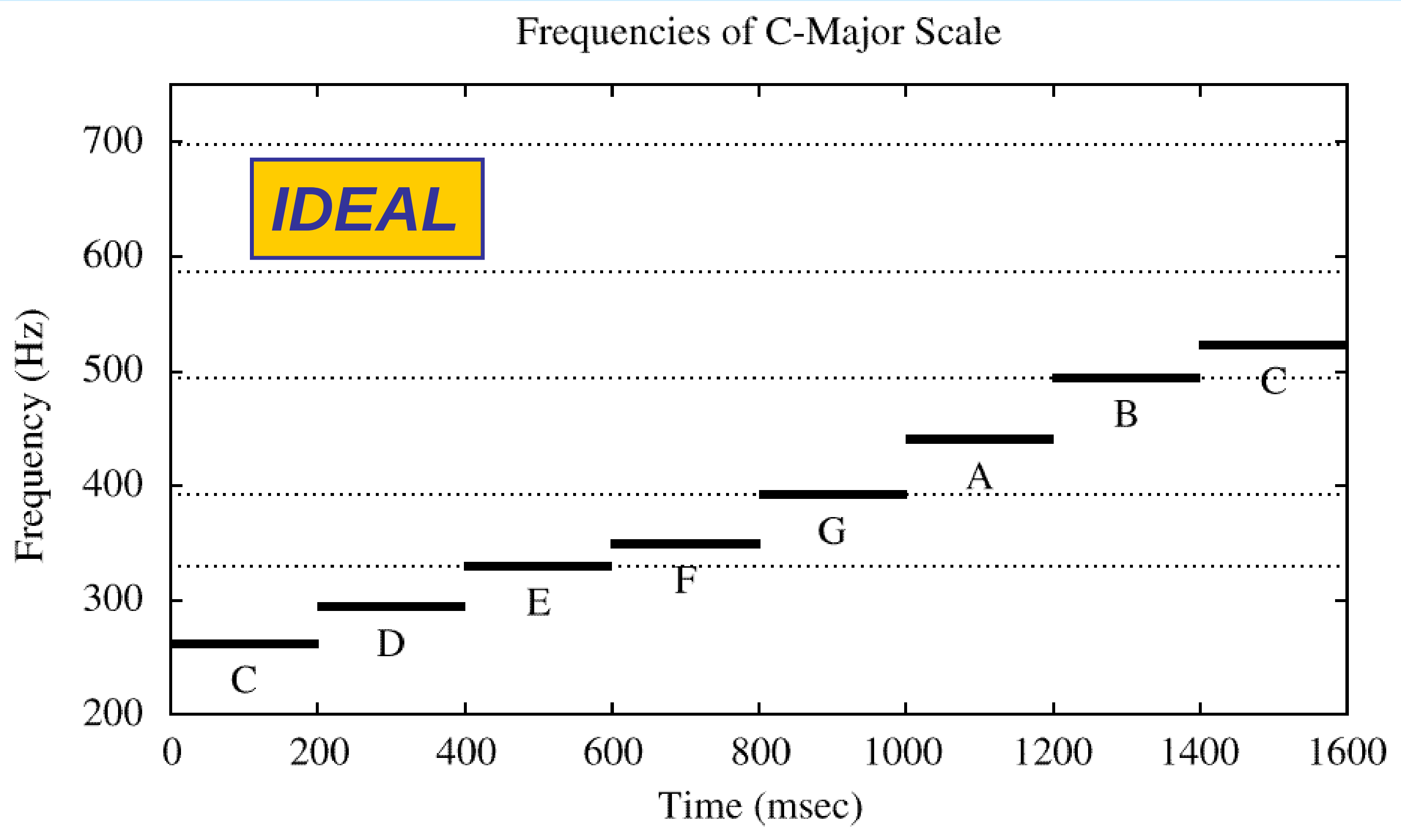


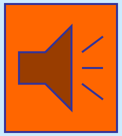
What is the fundamental frequency?

648 Hz ? **24 Hz ?**

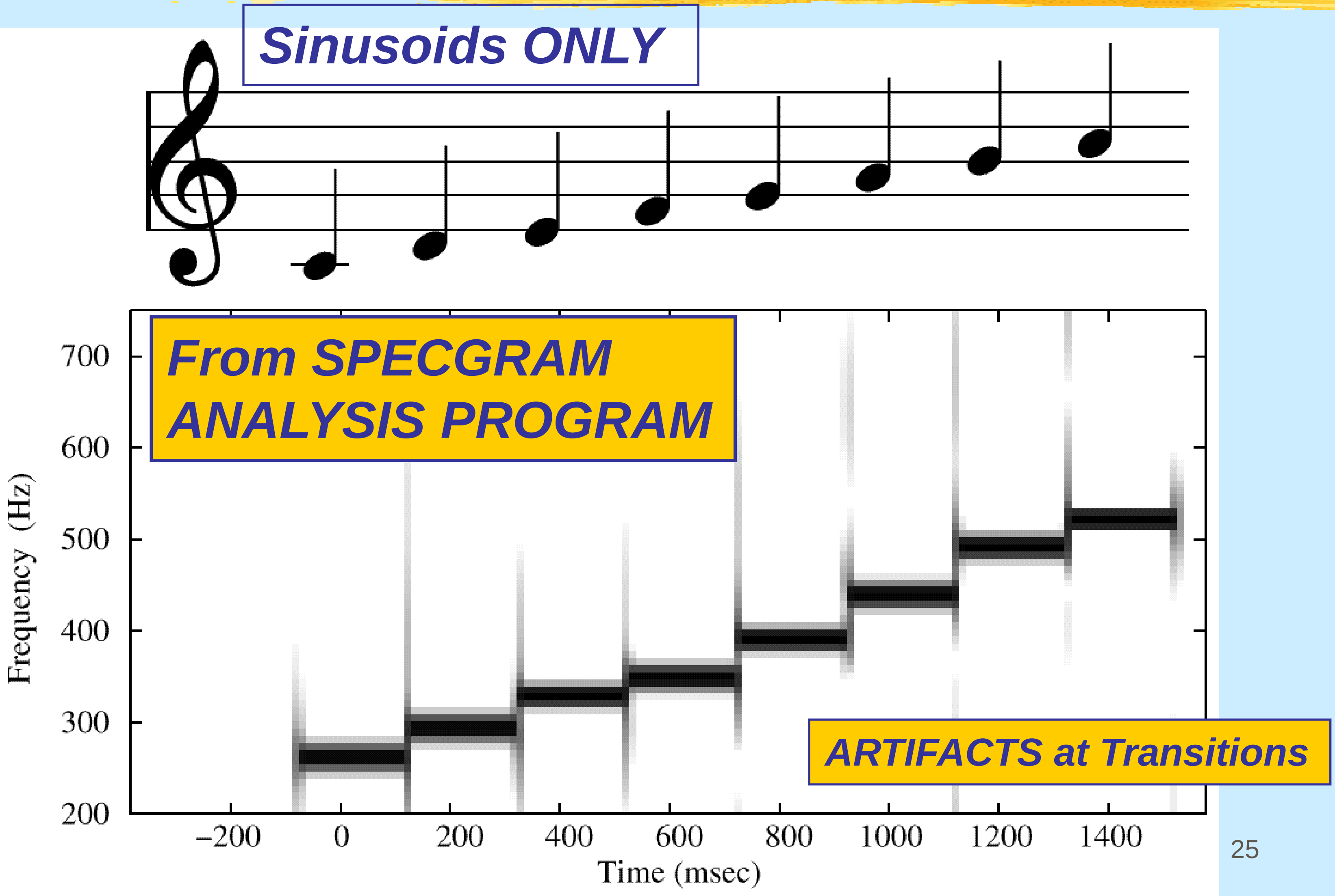
STEPPED FREQUENCIES

- C-major SCALE: successive sinusoids
  - Frequency is constant for each note

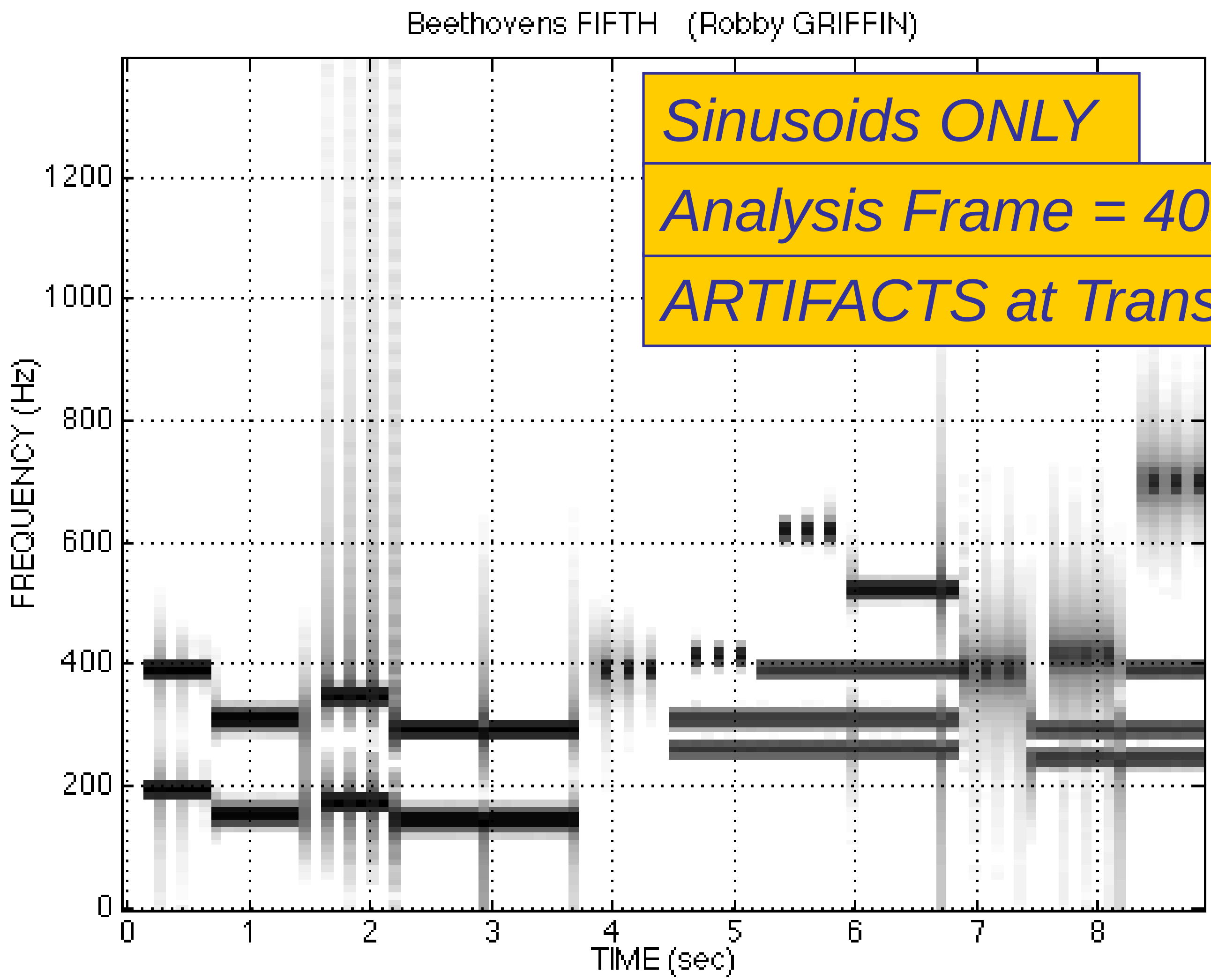




SPECTROGRAM of C-Scale



Spectrogram of LAB SONG



Sinusoids ONLY

Analysis Frame = 40ms

ARTIFACTS at Transitions

Time-Varying Frequency

- Frequency can change vs. time
  - Continuously, not stepped
- **<u>FREQUENCY MODULATION (FM)</u>**

$$x(t) = \cos(2\pi f_c t + v(t))$$

VOICE

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)

New Signal: Linear FM

- Called Chirp Signals (LFM)
  - Quadratic phase

QUADRATIC (pointing to t^2)

$$x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \varphi)$$

- Freq will change LINEARLY vs. time
  - Example of Frequency Modulation (FM)
  - Define "instantaneous frequency"

INSTANTANEOUS FREQ

- Definition

$$x(t) = A\cos(\psi(t))$$
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t)$$

Derivative of the "Angle"

- For Sinusoid:

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$
$$\psi(t) = 2\pi f_0 t + \varphi$$
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\pi f_0$$

Makes sense

INSTANTANEOUS FREQ of the Chirp

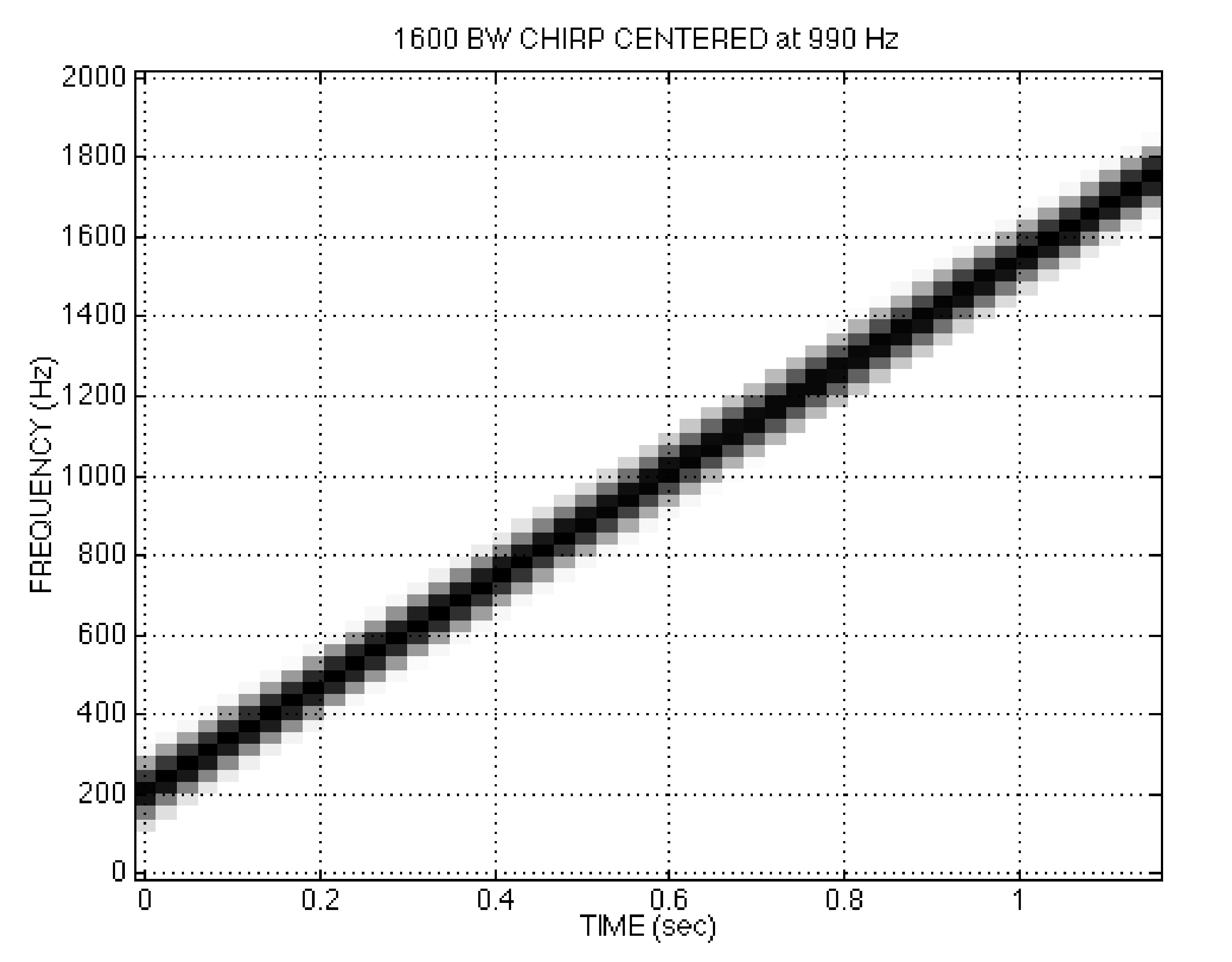
- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$

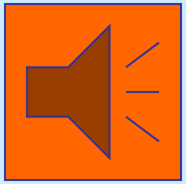
$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$

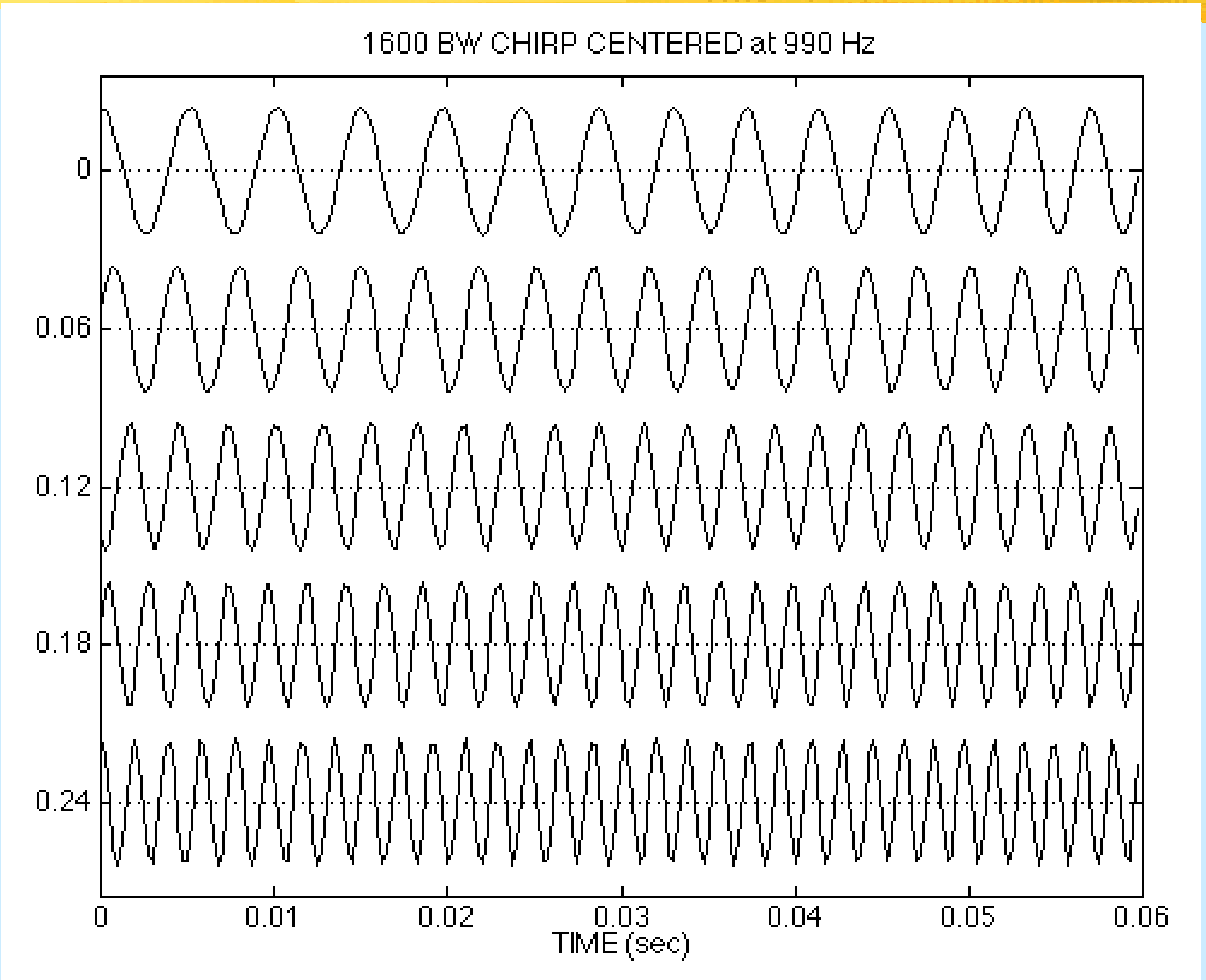
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\alpha t + \beta$$

CHIRP SPECTROGRAM



CHIRP WAVEFORM





OTHER CHIRPS

- $\psi(t)$ can be anything:

$$x(t) = A\cos(\alpha\cos(\beta t) + \varphi)$$

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = -\alpha\beta\sin(\beta t)$$

- $\psi(t)$ could be speech or music:
  - FM radio broadcast

SINE-WAVE FREQUENCY MODULATION (FM)