

Signal Processing First



Lecture 6

Fourier Series Coefficients

READING ASSIGNMENTS



- This Lecture:
 - **Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
 - Replaces pp. 62-66 in Ch 3 in DSP First
 - Notation: \mathbf{a}_k for Fourier Series
- Other Reading:
 - Next Lecture: More Fourier Series

LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k / T_0)t} dt$$

- ANALYSIS via Fourier Series
 - For PERIODIC signals: $\mathbf{x(t+T_0) = x(t)}$
 - Later: spectrum from the Fourier Series

HISTORY



- Jean Baptiste Joseph Fourier
 - 1807 thesis (memoir)
 - On the Propagation of Heat in Solid Bodies
 - Heat !
 - Napoleonic era

- <http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html>



Joseph Fourier

lived from 1768 to 1830

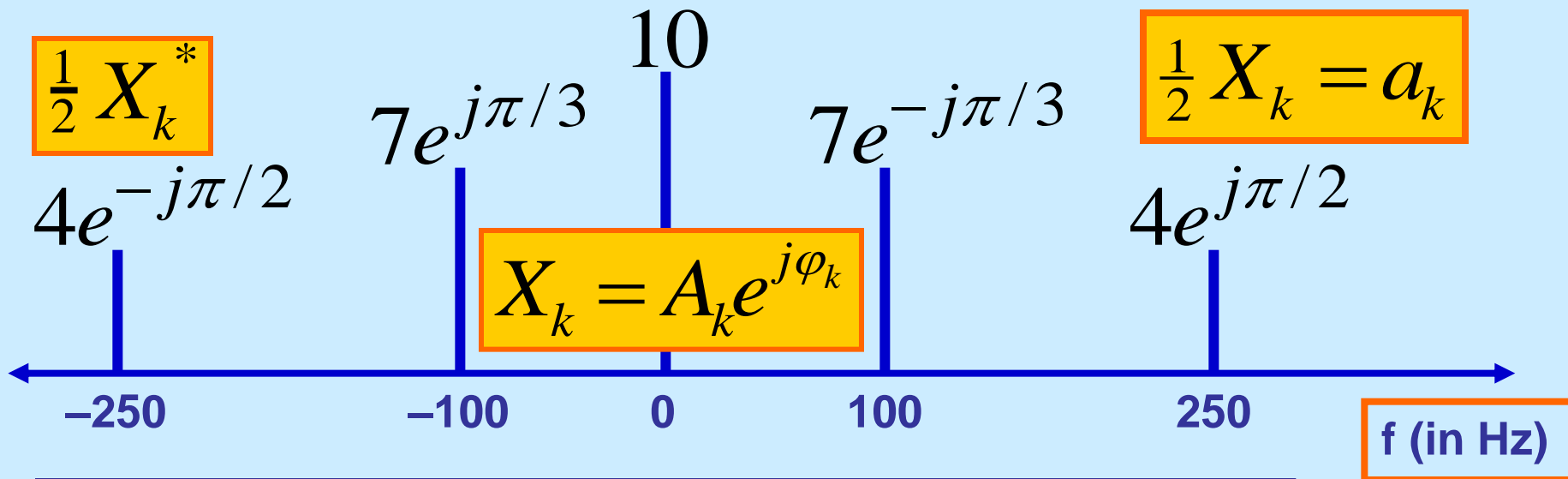
Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html>

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^N \left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

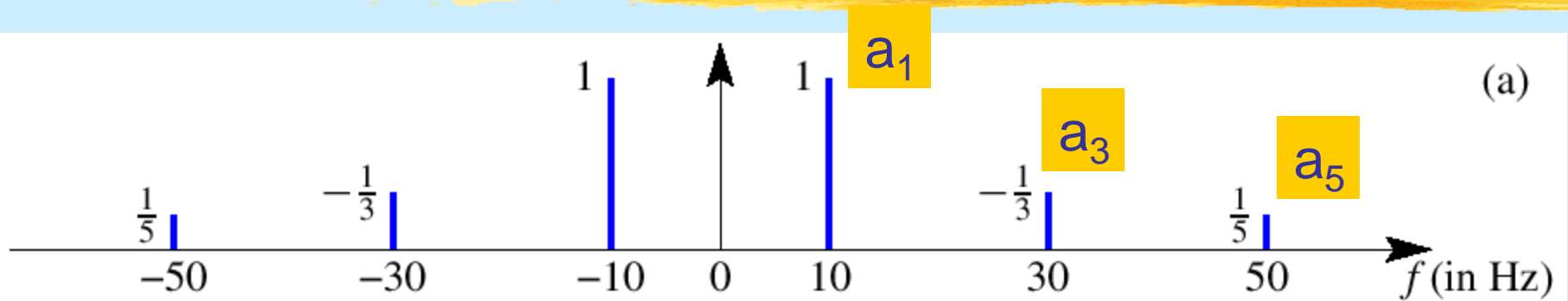
$$a_k = \frac{1}{2} X_k = \frac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

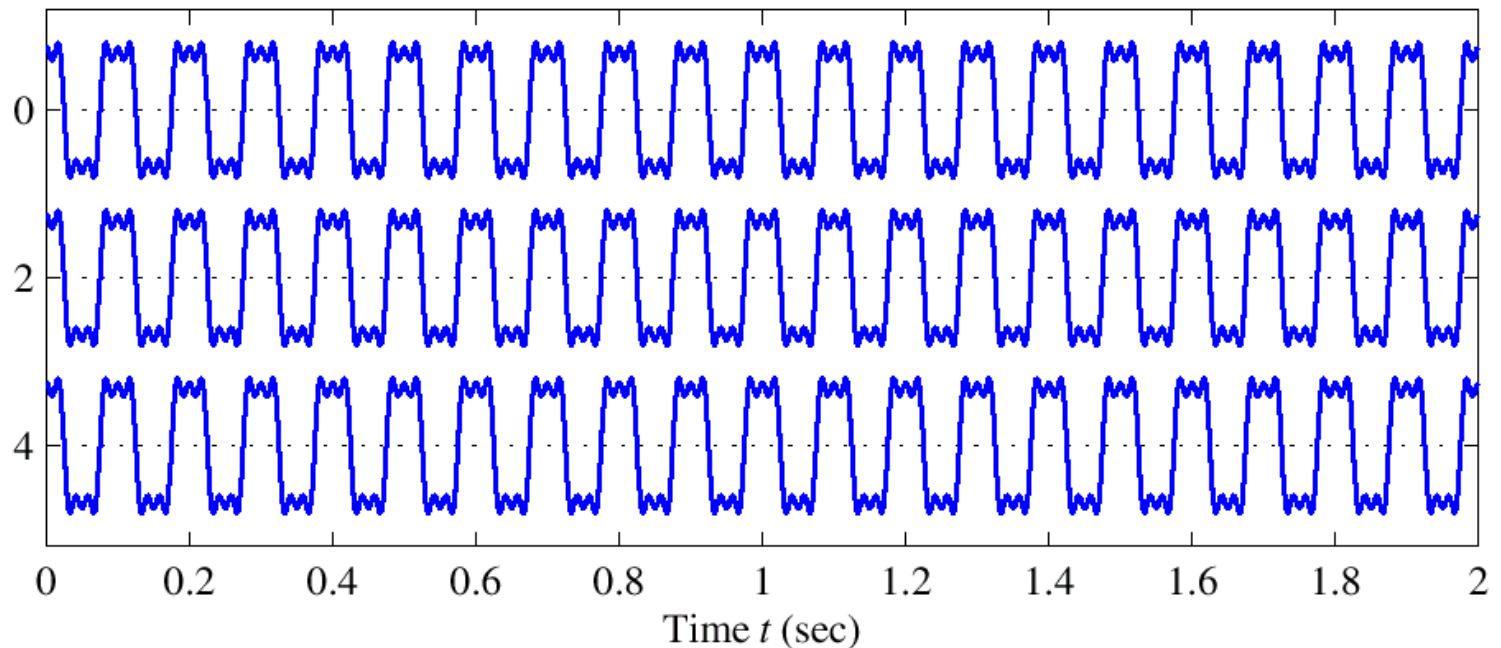
COMPLEX
AMPLITUDE

Harmonic Signal (3 Freqs)



$T = 0.1$

Sum of Cosine Waves with Harmonic Frequencies



SYNTHESIS vs. ANALYSIS

■ SYNTHESIS

- **Easy**
- Given (ω_k, A_k, ϕ_k) create $x(t)$
- Synthesis can be **HARD**
 - Synthesize Speech so that it sounds good

■ ANALYSIS

- **Hard**
- Given $x(t)$, extract (ω_k, A_k, ϕ_k)
- How many?
- Need algorithm for computer

STRATEGY: $x(t) \rightarrow a_k$

■ ANALYSIS

- Get representation from the signal
- Works for PERIODIC Signals
- Fourier Series
 - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

INTEGRAL Property of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt} \Big|_0^{T_0}$$
$$= \frac{T_0}{-j2\pi m} (e^{-j2\pi m} - 1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = 0$$

$$m \neq 0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of $\exp(j)$

- PRODUCT of $\exp(+j)$ and $\exp(-j)$

$$\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt} dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t} dt$$

Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t} dt = \frac{1}{T_0} \int_0^{T_0} \left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \right) e^{-j(2\pi/T_0)\ell t} dt$$

$$\frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t} dt = \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)kt} e^{-j(2\pi/T_0)\ell t} dt \right) = a_\ell$$

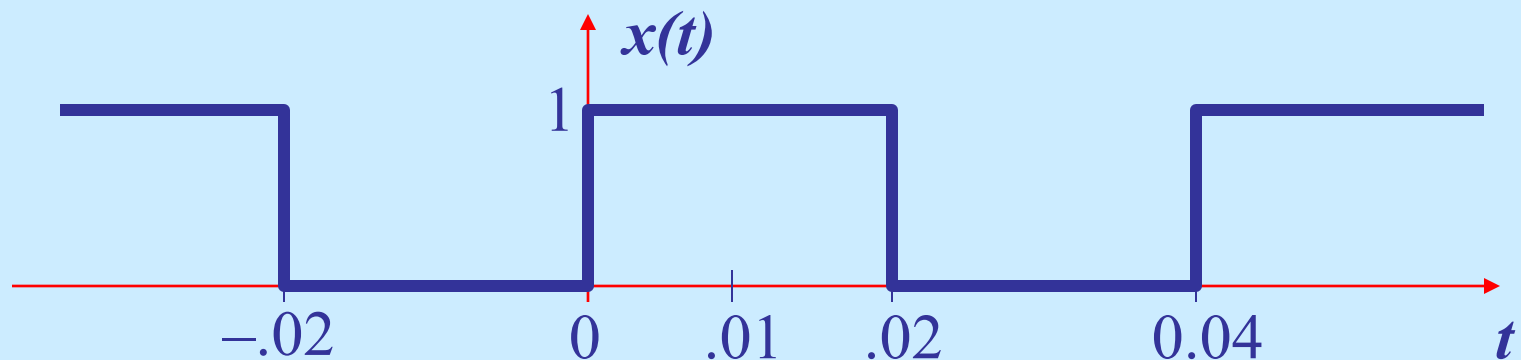
$$\Rightarrow a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Integral is zero
except for $k = \ell$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} T_0 \\ 0 & \frac{1}{2} T_0 \leq t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq 0)$$

$$a_k = \frac{1}{.04} \int_0^{.02} 1 e^{-j(2\pi/.04)kt} dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt} \Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)} (e^{-j(\pi)k} - 1) = \frac{1 - (-1)^k}{j2\pi k}$$

DC Coefficient: a_0

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k = 0)$$

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} (\text{Area})$$

$$a_0 = \frac{1}{.04} \int_0^{.02} 1 dt = \frac{1}{.04} (.02 - 0) = \frac{1}{2}$$

Fourier Coefficients a_k

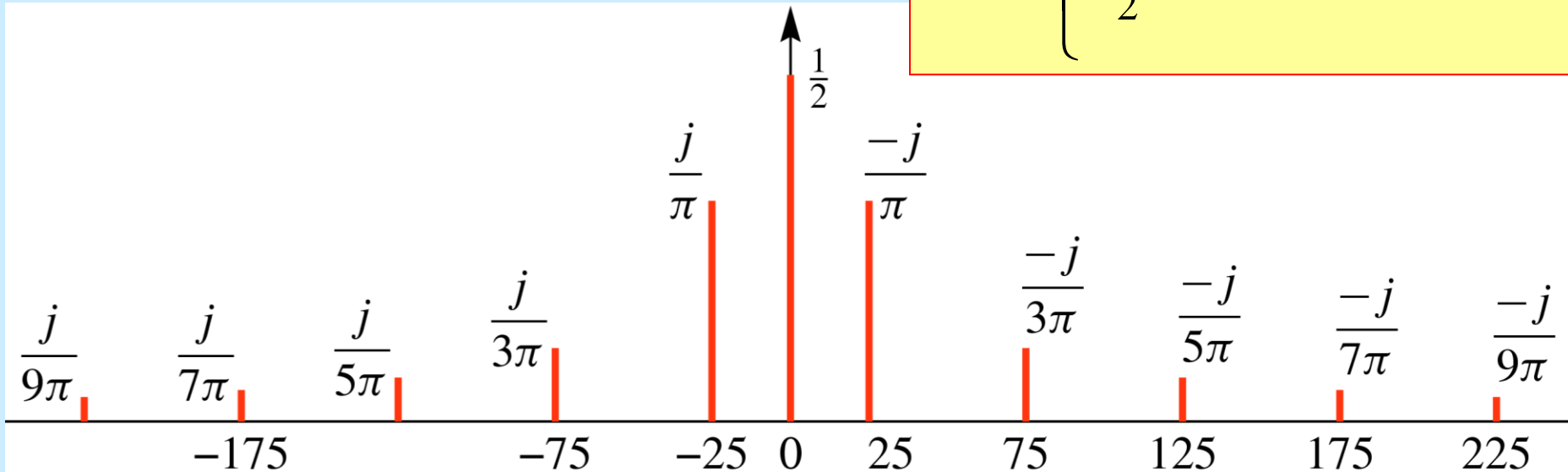
- a_k is a function of k
 - Complex Amplitude for k -th Harmonic
 - This one doesn't depend on the period, T_0

$$a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Integral

- HOW do you determine a_k from $x(t)$?

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Fundamental Frequency $f_0 = 1/T_0$

$a_{-k} = a_k^*$ when $x(t)$ is real

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt \quad (\text{DC component})$$