3.2 SEQUENCES AND SUMMATIONS

DEF: A ***sequence in a set*** A is a function f from a subset of the integers (usually $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$) to A. The values of a sequence are also called ***terms*** or ***entries***.

NOTATION: The value $f(n)$ is usually denoted a_n. A sequence is often written $a_0, a_1, a_2, \ldots$.

Example 3.2.1: Two sequences.

$$a_n = \frac{1}{n} \qquad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$
$$b_n = (-1)^n \qquad 1, -1, 1, -1, \ldots$$

Example 3.2.2: Five ubiquitous sequences.

$$n^2 \quad 0, 1, 4, 9, 16, 25, 36, 49, \ldots$$
$$n^3 \quad 0, 1, 8, 27, 64, 125, 216, 343, \ldots$$
$$2^n \quad 1, 2, 4, 8, 16, 32, 64, 128, \ldots$$
$$3^n \quad 1, 3, 9, 27, 81, 243, 729, 2187, \ldots$$
$$n! \quad 1, 1, 2, 6, 24, 120, 720, 5040, \ldots$$

STRINGS

DEF: A set of characters is called an ***alphabet***.

Example 3.2.3: Some common alphabets:

$\{0, 1\}$ the binary alphabet

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ the decimal digits

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$
the hexadecimal digits

$\{A, B, C, D, \ldots, X, Y, Z\}$ English uppercase

ASCII

DEF: A ***string*** is a sequence in an alphabet.

NOTATION: Usually a string is written without commas, so that consecutive characters are juxtaposed.

Example 3.2.4: If $f(0) = M, f(1) = A$, $f(2) = T,$ and $f(3) = H$, then write "MATH".

SPECIFYING a RULE

Problem: Given some initial terms $a_0, a_1, ..., a_k$ of a sequence, try to construct a rule that is consistent with those initial terms.

Approaches: There are two standard kinds of rule for calculating a generic term a_n.

DEF: A ***recursion*** for a_n is a function whose arguments are earlier terms in the sequence.

DEF: A ***closed form*** for a_n is a formula whose argument is the subscript n.

Example 3.2.5: $1, 3, 5, 7, 9, 11, \ldots$

recursion: $a_0 = 1; \quad a_n = a_{n-1} + 2$ for $n \geq 1$

closed form: $a_n = 2n + 1$

The differences between consecutive terms often suggest a recursion. Finding a recursion is usually easier than finding a closed formula.

Example 3.2.6: $1, 3, 7, 13, 21, 31, 43, \ldots$

recursion: $b_0 = 1; \quad b_n = b_{n-1} + 2n$ for $n \geq 1$

closed form: $b_n = n^2 + n + 1$

Sometimes, it is significantly harder to construct a closed formula.

Example 3.2.7: $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

recursion: $c_0 = 1, c_1 = 1;$
$c_n = c_{n-1} + c_{n-2}$ for $n \geq 1$

closed form: $c_n = \frac{1}{\sqrt{5}}\left[G^{m+1} - g^{m+1}\right]$
where $G = \frac{1+\sqrt{5}}{2}$ and $g = \frac{1-\sqrt{5}}{2}$

INFERRING a RULE

The ESSENCE of science is inferring rules from partial data.

Example 3.2.8: Sit under apple tree. Infer gravity.

Example 3.2.9: Watch starlight move 0.15 arc-seconds in total eclipse. Infer relativity.

Example 3.2.10: Observe biological species. Infer DNA.

Important life skill: Given a difficult general problem, start with special cases you can solve.

Example 3.2.11: Find a recursion and a closed form for the arithmetic progression:

$$c, c+d, c+2d, c+3d, \ldots$$

recursion: $a_0 = c; \quad a_n = a_{n-1} + d$

closed form: $a_n = c + nd$.

Q: How would you decide that a given sequence is an arithmetic progression?

A: Calculate differences betw consec terms.

DEF: The ***difference sequence*** for a sequence a_n is the sequence $a'_n = a_n - a_{n-1}$ for $n \geq 1$.

Example 3.2.5 redux:

a_n :	1	3	5	7	9	11
a'_n :	2	2	2	2	2	

Analysis: Since a'_n is constant, the sequence is specified by this recursion:

$$a_0 = 1; a_n = a_{n-1} + 2 \text{ for } n \geq 1.$$

Moreover, it has this closed form:

$$\begin{aligned} a_n &= a_0 + a'_1 + a'_2 + \cdots + a'_n \\ &= a_0 + 2 + 2 + \cdots + 2 \quad = 1 + 2n \end{aligned}$$

If you don't get a constant sequence on the first difference, then try reiterating.

Revisit Example 1.7.6: $1, 3, 7, 13, 21, 31, 43, \ldots$

b_n :	1	3	7	13	21	31	43
b'_n :	2	4	6	8	10	12	
b''_n :	2	2	2	2	2		

Analysis: Since b''_n is constant, we have

$b'_n = 2 + 2n$

Therefore,

$$\begin{aligned} b_n &= b_0 + b'_1 + b'_2 + \cdots + b'_n \\ &= b_0 + 2\sum_{j=1}^{n} j = 1 + (n^2 + n) = n^2 + n + 1 \end{aligned}$$

Consolation Prize: Without knowing about finite sums, you can still extend the sequence:

b_n :	1	3	7	13	21	31	43	$\underline{57}$
b'_n :	2	4	6	8	10	12	$\underline{14}$	
b''_n :	2	2	2	2	2	$\underline{2}$		

SUMMATIONS

DEF: Let a_n be a sequence. Then the ***big-sigma*** notation

$$\sum_{j=m}^{n} a_j$$

means the sum

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

TERMINOLOGY: j is the ***index of summation***

TERMINOLOGY: m is the ***lower limit***

TERMINOLOGY: n is the ***upper limit***

TERMINOLOGY: a_j is the ***summand***

Theorem 3.2.1. *These formulas for summing falling powers are provable by induction (see §3.3):*

$$\sum_{j=1}^{n} j^{\underline{1}} = \frac{1}{2}(n+1)^{\underline{2}} \qquad \sum_{j=1}^{n} j^{\underline{2}} = \frac{1}{3}(n+1)^{\underline{3}}$$

$$\sum_{j=1}^{n} j^{\underline{3}} = \frac{1}{4}(n+1)^{\underline{4}} \qquad \sum_{j=1}^{n} j^{\underline{k}} = \frac{1}{k+1}(n+1)^{\underline{k+1}}$$

Example 3.2.12: True Love and Thm 3.2.1

On the j^{th} day ... True Love gave me

$$j + (j-1) + \cdots + 1 = \frac{(j+1)^{\underline{2}}}{2} \text{ gifts.}$$

$$= \frac{1}{2}\sum_{j=2}^{13} j^{\underline{2}} = \frac{1}{2}\left[2^{\underline{2}} + \cdots + 13^{\underline{2}}\right]$$

$$= \frac{1}{2}\left[2 + 6 + \cdots + 78\right] = 364 \quad \text{slow}$$

$$= \frac{1}{2} \cdot \frac{14^{\underline{3}}}{3} = 364 \quad \text{fast}$$

Corollary 3.2.2. *High-powered look-ahead to formulas for summing* $j^k : j = 0, 1, ..., n$.

$$\sum_{j=1}^{n} j^2 = \sum_{j=1}^{n}(j^{\underline{2}} + j^{\underline{1}}) = \frac{1}{3}(n+1)^{\underline{3}} + \frac{1}{2}(n+1)^{\underline{2}}$$

$$\sum_{j=1}^{n} j^3 = \sum_{j=1}^{n}(j^{\underline{3}} + 3j^{\underline{2}} + j^{\underline{1}}) = \cdots$$

POTLATCH RULES for CARDINALITY

DEF: ***nondominating cardinality:*** Let A and B be sets. Then $|A| \leq |B|$ means that $\exists$ one-to-one function $f : A \to B$.

DEF: Set A and B have ***equal cardinality*** (write $|A| = |B|$) if $\exists$ bijection $f : A \to B$, which obviously implies that $|A| \leq |B|$ and $|B| \leq |A|$.

DEF: ***strictly dominating cardinality:*** Let A and B be sets. Then $|A| < |B|$ means that $|A| \leq |B|$ and $|A| \neq |B|$.

DEF: The ***cardinality*** of a set A is n if $|A| = |\{1, 2, \ldots, n\}|$ and 0 if $A = \emptyset$. Such cardinalities are called ***finite***. NOTATION: $|A| = n$.

DEF: The ***cardinality*** of $\mathcal{N}$ is ω ("omega"), or alternatively, $\aleph_0$ ("aleph null").

DEF: A set is ***countable*** if it is finite or ω.

Remark: $\aleph_0$ is the smallest infinite cardinality. The real numbers have cardinality $\aleph_1$ ("aleph one"), which is larger than $\aleph_0$, for reasons to be given.

INFINITE CARDINALITIES

Proposition 3.2.3. *There are as many even nonnegative numbers as non-negative numbers.*

Proof: $f(2n) = n$ is a bijection. ◊

Theorem 3.2.4. *There are as many positive integers as rational fractions.*

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Proof: $f\left(\frac{p}{q}\right) = \frac{(p+q-1)(p+q-2)}{2} + p$ ◊

Example 3.2.13: $f\left(\frac{2}{3}\right) = \frac{(4)(3)}{2} + 2 = 8$

Theorem 3.2.5. *(G. Cantor) There are more positive real numbers than positive integers.*

Semi-proof: A putative bijection $f : \mathcal{Z}^+ \to \mathcal{R}^+$ would generate a sequence in which each real number appears somewhere as an infinite decimal fraction, like this:

$$\begin{aligned} f(1) &= .\underline{8}841752032669031\ldots \\ f(2) &= .1\underline{4}15926531424450\ldots \\ f(3) &= .32\underline{0}2313932614203\ldots \\ f(4) &= .167\underline{9}888138381728\ldots \\ f(5) &= .0452\underline{9}98136712310\ldots \\ &\ldots \\ f(?) &= .73988\ldots \end{aligned}$$

Let $f(n)_k$ be the kth digit of $f(n)$, and let π be the permutation $0 \mapsto 9, 1 \mapsto 0, \ldots 9 \mapsto 8$. Then the infinite decimal fraction whose kth digit is $\pi(f(n)_k)$ is not in the sequence. Therefore, the function f is not onto, and accordingly, not a bijection. ♢