6.1 RECURRENCE RELATIONS

DEF: A ***recurrence system*** is a finite set of ***initial conditions***

$$a_0 = c_0, \quad a_1 = c_1, \quad \ldots, \quad a_d = c_d$$

and a formula (called a ***recurrence relation***)

$$a_n = f(a_0, \ldots, a_{n-1})$$

that expresses a subscripted variable as a function of lower-indexed values. A sequence

$$< a_n > \quad = \quad a_0, a_1, a_2, \ldots$$

satisfying the initial conditions and the recurrence relation is called a ***solution***.

Example 6.1.1: The recurrence system with initial condition

$$a_0 = 0$$

and recurrence relation

$$a_n \quad = \quad a_{n-1} + 2n - 1$$

has the sequence of squares as its solution:

$$< a_n > \quad = \quad 0, 1, 4, 9, 16, 25, \ldots$$

NÄIVE METHOD OF SOLUTION

Step 1. Use the recurrence to calculate a few more values beyond the given initial values.

Step 2. Spot a pattern and guess the right answer.

Step 3. Prove your answer is correct (by induction).

Example 6.1.1, continued:

Step 1. Starting from $a_0 = 0$, we calculate

$$\begin{aligned} a_1 &= a_0 + 2 \cdot 1 - 1 &= 0 + 1 = 1 \\ a_2 &= a_1 + 2 \cdot 2 - 1 &= 1 + 3 = 4 \\ a_1 &= a_0 + 2 \cdot 3 - 1 &= 4 + 5 = 9 \\ a_1 &= a_0 + 2 \cdot 4 - 1 &= 9 + 7 = 16 \end{aligned}$$

Step 2. Looks like $f(n) = n^2$.

Step 3. BASIS: $a_0 = 0 = 0^2 = f(0)$.
IND HYP: Assume that $a_{n-1} = (n-1)^2$.
IND STEP: Then

$$\begin{aligned} a_n &= a_{n-1} + 2n - 1 \quad \text{from the recursion} \\ &= (n-1)^2 + 2n - 1 \quad \text{by IND HYP} \\ &= (n^2 - 2n + 1) + 2n - 1 \quad = \quad n^2 \quad \diamondsuit \end{aligned}$$

APPLICATIONS

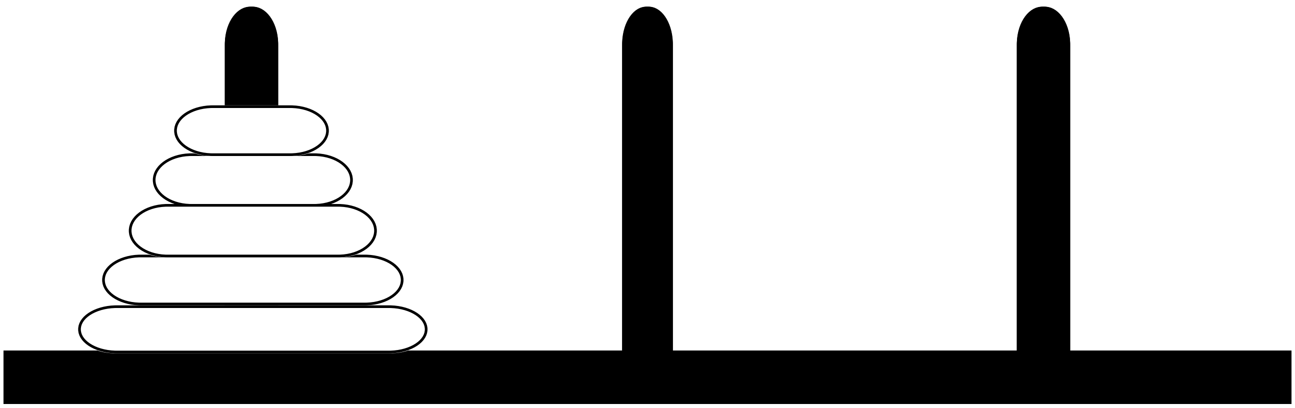
Example 6.1.2: Compound Interest
Deposit \$1 to compound at annual rate r.
$p_0 = 1 \quad p_n = (1+r)p_{n-1}$

EARLY TERMS: $1, 1+r, (1+r)^2, (1+r)^3, \ldots$
APPARENT PATTERN: $p_n = (1+r)^n$

BASIS: True for n = 0.
IND HYP: Assume that $p_{n-1} = (1+r)^{n-1}$
IND STEP: Then

$$\begin{aligned} p_n &= (1+r)p_{n-1} \quad \text{by the recursion} \\ &= (1+r)(1+r)^{n-1} \quad \text{by IND HYP} \\ &= (1+r)^n \quad \text{by arithmetic} \quad \diamondsuit \end{aligned}$$

Example 6.1.3: Tower of Hanoi



RECURRENCE SYSTEM
$h_0 = 0$
$h_n = 2h_{n-1} + 1$

SMALL CASES: $0, 1, 3, 7, 15, 31, \ldots$

APPARENT PATTERN: $h_n = 2^n - 1$

BASIS: $h_0 = 0 = 2^0 - 1$
IND HYP: Assume that $h_{n-1} = 2^{n-1} - 1$

IND STEP: Then

$$\begin{aligned} h_n &= 2h_{n-1} + 1 \quad \text{by the recursion} \\ &= 2(2^{n-1} - 1) + 1 \quad \text{by IND HYP} \\ &= 2^n - 1 \quad \text{by arithmetic} \quad \diamondsuit \end{aligned}$$

However, the näive method has limitations:
• It can be non-trivial to spot the pattern.
• It can be non-trivial to prove that the apparent pattern is correct.

Example 6.1.4: Fibonacci Numbers
$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$

Fibo seq: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$.

APPARENT PATTERN (ha ha)

$$f_n = \frac{1}{2^n\sqrt{5}}\left[(1+\sqrt{5})^n - (1-\sqrt{5})^n\right]$$

It is possible, but not uncomplicated, to simplify this with the binomial expansion and to then use induction.

Sometimes there is no fixed limit on the number of previous terms used by a recursion.

Example 6.1.5: Catalan Recursion

$c_0 = 1$

$c_n = c_0 c_{n-1} + c_1 c_{n-2} + \cdots + c_{n-1} c_0$ for $n \geq 1$.

SMALL CASES

$$c_1 = c_0 c_0 = 1 \cdot 1 = 1$$

$$c_2 = c_0 c_1 + c_1 c_0 = 1 \cdot 1 + 1 \cdot 1 = 2$$

$$c_3 = c_0 c_2 + c_1 c_1 + c_2 c_0 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$$

$$c_4 = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14$$

$$c_5 = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42$$

Catalan seq: $1, 1, 2, 5, 14, 42, \ldots$.

SOLUTION: $c_n = \dfrac{1}{n+1}\dbinom{2n}{n}$

The Catalan recursion counts binary trees and other objects in computer science.

ADMONITION

- Most recurrence relations have no solution.
- Most sequences have no representation as a recurrence relation. (they are random)